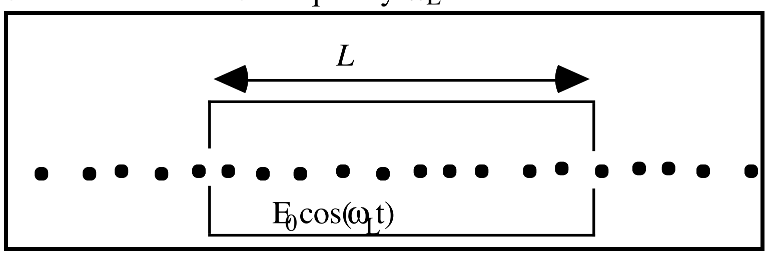
Physics 566 Problem Set #2

Due: Friday Sep. 10, 2010

Problem 1: Magnetic Resonance: Rabi vs. Ramsey (20 Points)

The technique of measuring transition frequencies with magnetic resonance was pioneered by I. I. Rabi in the late 30's. It was modified by Ramsey (his student) about 10 years later, and now serves as the basis for atomic clocks and the SI definition of the second. All precision atomic measurements, including modern atom-interferometers and quantum logic gates in atomic systems, have at their heart a Ramsey type geometry.

(i) Rabi resonance geometry. Consider a beam of two-level atoms with transition frequency ω_{eg} , passing through an "interaction zone" of length *L*, in which they interact with a monochromatic laser field of frequency ω_L.

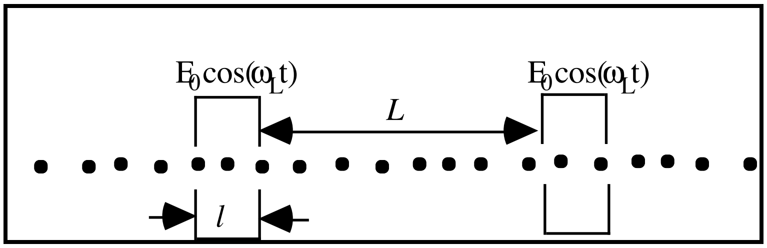


(a) Suppose all the atoms start in the ground-state $|g\rangle$, and have a well defined velocity v, chosen such that $\Omega L / v = \pi$, where $\hbar\Omega = d_{eg}E_0$. Plot the probability to be in the excited state $|e\rangle$, P_e, as a function of driving frequency ω_L, neglecting spontaneous emission (what is the condition that we can do this?). What is the linewidth? Explain your plot in terms of the Bloch-sphere.

(b) Now suppose the atoms have a distribution of velocities characteristic of thermal beams: $f(v) = \frac{2}{v_0^4} v^3 \exp(-v^2 / v_0^2)$, where $v_0 = \sqrt{2k_B T / m}$. Plot P_e vs. *L* for Δ=0, (you may need to do this numerically). At what *L* is it maximized - explain? Also plot as in (a), P_e as a function of ω_L with $L = L_{max}$. What is the linewidth? Explain in terms of the Bloch-sphere.

(ii) Ramsey separated zone method

As you have seen in parts (a)-(b), assuming one can make the velocity spread sufficiently small, the resonance linewidth is limited by the interaction time *L*/*v*. This is known as "transit-time broadening" and is a statement of the time-energy uncertainty principle. Unfortunately, if we make *L* larger and larger other inhomogenities, such as the amplitude of the driving field come into play. Ramsey's insight was that one can in fact "break up" the π-pulse given to the atoms into two π/2-pulses in a time $\tau = l/v$ (i.e. $\Omega\tau = \pi/2$), separated by *no interaction* for a time $T = L/v$. The free interaction time can then made *much* longer.



(b) Given a mono-energetic atoms with velocity v, internal state $|\psi(0)\rangle = |g\rangle$, and field at a detuning $|\Delta| << \Omega$ so that $\tilde{\Omega} \approx \Omega$ find:

$$|\psi(\tau = l/v)\rangle,\ |\psi(\tau + T = (l+L)/v)\rangle,\ |\psi(2\tau + T = (2l + L)/v)\rangle$$

and show that mapping of the state on the Bloch-sphere.

(c) Plot $P_e(t_{final} = 2\tau + T)$ as a function of ω_L. Plot also for the case of finite spread in velocity as in (b). What is the linewidth?

(d) A Ramsey separated zone geometry is often described as a kind of " interferometer". Explain why this makes sense.

Problem 2: Spin precession in a magnetic field - Heisenberg picture (10 Points)

Consider a spin 1/2 particle such as an electron in a magnetic field. Such a particle has an intrinsic magnetic moment, described by the operator, $\hat{\vec{\mu}} = \gamma_s \hat{\mathbf{S}}$, where γ_S is known as the "gyromagnetic ratio", and $\mathbf{S} = \hat{S}_x\mathbf{e}_x + \hat{S}_y\mathbf{e}_y + \hat{S}_z\mathbf{e}_z$ in the spin 1/2 angular momentum operator. When placed in a magnetic field **B**, the interaction energy is described by the Hamiltonian

$$\hat{H} = -\hat{\vec{\mu}} \cdot \mathbf{B}\ .$$

(a) Show that the Heisenberg equation of motion for the spin operator is

$$\frac{d\hat{\mathbf{S}}}{dt} = -\vec{\Omega} \times \hat{\mathbf{S}}\text{, where } \vec{\Omega} = \gamma_s \mathbf{B}$$

Describe the physical meaning of this differential equation if we take **S** to be a classical angular momentum vector.

(b) Find the Heisenberg equations of motion for the spherical components $\hat{\sigma}_z, \hat{\sigma}_\pm$ (do this through direct commutation with the Hamiltonian and check with part (a)).

(c) Solve this equation for $\hat{\sigma}_x(t), \hat{\sigma}_y(t), \hat{\sigma}_z(t)$ in terms of the initial operators for the particular case the magnetic field is $\mathbf{B} = B_x\mathbf{e}_x + B_z\mathbf{e}_z$. Use this solution to find the trajectory of the Bloch vector $\mathbf{Q}(t) = \langle\vec{\sigma}(t)\rangle$ for the Heisenberg state $|-\rangle_z$ (this is the initial state in the Schrödinger picture). Sketch the trajectory on the Bloch sphere.

Problem 3: Inhomogeneous broadening

In a typical experiment to observe quantum coherence, one is dealing with an *ensemble* of systems, e.g. atoms in a gas, molecules in a solvent, water in human tissue. The signal one observes is then a sum over all of the members. Because these ensembles are extended in space, they tend to be exposed to inhomogeneous environments. All members are thus not undergoing identical evolutions and we must average some distribution function. The result can lead to a "washing out" of coherent oscillations in the signal. This is NOT however due to a loss of quantum coherence. Through clever techniques one can recover the coherent oscillations an thwart the effects of inhomogeneity.

(a) <u>Free induction decay by inhomogeneous broadening:</u> Consider a macroscopic ensemble of spins in static magnetic field in the z direction, but with an inhomogeneous magnitude. If a given spin starts in $|+_x\rangle$ and sees a magnetic field $B_\parallel$, it will Larmor precess at $\Omega_\parallel = \gamma B_\parallel$ (γ being the gyromagnetic ratio). This is known as "free induction" as the oscillating spin will radiate as freely rotating magnetic dipole. This signal will decay due to the averaging over different local precession frequencies, $\Omega_\parallel$.

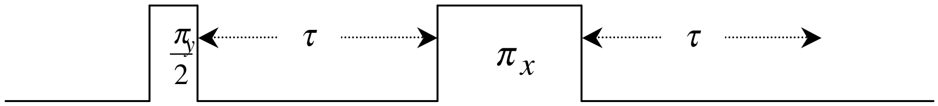
Suppose the distribution of $B_\parallel$ is Gaussian, with mean B_0 and rms $\Delta B_\parallel << B_0$.

(i) Calculate and sketch, $\langle\hat{\sigma}_x\rangle$, averaged over the ensemble, as a function of time.

(ii) What the characteristic decay time, known as T_2^*, due to inhomogeneity.

(iii) If the spins all start in $|+_z\rangle$, qualitatively describe how to achieve an approximate $\pi/2$-pulse to rotate all spins into the state $|+_x\rangle$, despite the inhomogeneity.

(b) <u>Spin echo:</u> Though inhomogeneous broadening will cause a decay of the ensemble averaged coherence, it is not a truly irreversible process. A procedure for recovering the coherence is known as a "spin echo". Consider the following pulse sequence.



The $\pi/2$-pulse about the *y*-axis acts according to (a.iii) to bring all spins onto the *x*-axis of the Bloch sphere. For a time τ, the spins dephase. The π-pulse about the *x*-axis acts to time reverse the process. An "echo" signal will be seen at a time τ later.
Explain this process using this Bloch sphere. **Sketch** the signal $\langle\hat{\sigma}_x\rangle$.